Journal of Global Optimization **13:** 407–416, 1998.

A Simplified Convergence Proof for the Cone Partitioning Algorithm

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(Accepted in final form 30 June 1998)

Abstract. We present a new convergence result for the cone partitioning algorithm with a pure ω-subdivision strategy, for the minimization of a quasiconcave function over a polytope. It is shown that the algorithm is finite when ε-optimal solution with $\varepsilon > 0$ are looked for, and that any cluster point of the points generated by the algorithm is an optimal solution in the case $\varepsilon = 0$. This result improves on the one given previously by the authors, its proof is simpler and relies more directly on a new class of hyperplanes and its associated simplicial lower bound.

Key words: Cone partitioning, Convergence, ω-Subdivision, (Quasi)concave minimization

1. Introduction

We consider the following quasiconcave programming problem

$$(CP) \quad \min\{f(x) \mid x \in P\}$$

where f is a quasiconcave continuous function on $\mathbb{R}^n$ and P is a full dimensional polytope of $\mathbb{R}^n$.

Tuy [11] proposed in 1964 a first method, a cone covering algorithm, to solve the concave programming problem. Unfortunately this method was shown to cycle on some problems by Zwart [14] (recently Meyer [9] has shown that this cycling does not prevent the algorithm from finding an optimal solution of the problem). Independently, Bali [1] and Zwart [15] proposed to consider partitioning of cones instead of covering. The two algorithms differ from each other only when ε-optimal solution ($\varepsilon > 0$) are looked for, but are identical when optimal solution (i.e., with $\varepsilon = 0$) are sought. In particular, they both make use of ω-subdivisions, i.e., cones are subdivided with respect to a point ω of the polytope that is a byproduct of the deletion test. In 1981, Jacobsen [3] proposed a proof of Bali's algorithm convergence, but unfortunately this proof used a separation property which may not hold in the general case (see Tuy [12]).

In the meantime, the difficulty in proving the convergence of the conical algorithms was alleviated by modifying the subdivision process. In 1980, Thoai and Tuy [10] proposed to replace ω-subdivisions by bisections, and showed the convergence of the resulting algorithm. However, this strategy deteriorates the computational performances, due to the fact that the bisection process does not make use of the structure of the problem. This conflict between efficiency and convergence has been partially solved with the concept of *normal subdivision* introduced by Tuy [13], which consists in using ω-subdivisions most of the time, and bisections occasionally to ensure the convergence. A new proof of the convergence of the cone partitioning algorithm with a pure ω-subdivision strategy was first proposed in [4]. Since then, the nature of the hyperplane that plays a key role in the proof has been better understood and it has yield to a new simplicial lower bound that dominates the classical one (see Jaumard and Meyer [5] and Meyer [8]). In this paper, we give a stronger convergence result than in [4] and propose a shorter proof that takes into account the recent developments about the hyperplane.

The reader is referred to the book of Horst and Tuy [2] for more details on conical algorithms and for a description of other methods for solving problem (CP).

The paper is organized as follows. In Section 2, the basic operations for the cone partitioning algorithm are recalled. In Section 3, the algorithm is described. The convergence is discussed in Section 4. Conclusions are drawn in the last section.

2. Basic operations

In this section, we discuss the basic operations needed to define the algorithm, namely the construction of the initial conical partition (Section 2.1), the computation of γ-extensions (Section 2.2), the deletion test (Section 2.3) and the ω-subdivisions (Section 2.4).

2.1. INITIAL PARTITION

The initialization part consists in rewriting the problem (CP) in the following form

$$(CP') \quad \min\{f(x) \mid x \in K^0 \cap P'\}$$

where f is the quasiconcave function, K^0 is a polyhedral cone vertexed at O and with exactly n independent edges, and P' is a polyhedron of $\mathbb{R}^n$ containing O in its interior and such that $K^0 \cap P'$ is bounded.

This can be done in several ways:

1. Choose a non-degenerated vertex v^0 of P (assuming that there exists one) and perform a change of variable which transforms v^0 into O. The cone K^0 is defined by the constraints of P binding at v^0 and the polyhedron P' is obtained from P by deleting those constraints.
2. Perform a change of variable in such a way that O is an interior point of P. Then construct a partition of $\mathbb{R}^n$ into $n+1$ cones (see, e.g., Horst and Tuy [2]). We obtain $n+1$ problems of the form (CP') with $P' = P$ and where K^0 is successively each of the $n+1$ cones (these problems are usually solved simultaneously: see for example [4]).

In the first case, all optimal solutions of problem (CP') are also optimal solutions of problem (CP), while in the second case optimal solutions of at least one problem (CP') are optimal solutions of (CP).
From now on, we assume that $P' = \{x \in \mathbb{R}^n \mid A'x \leq b'\}$. Moreover, we will denote by $K = \text{cone}\{u^1, \ldots, u^n\}$ the polyhedral cone of origin O whose directions are the n linearly independent vectors $u^1, \ldots, u^n$.

2.2. γ-EXTENSIONS

In order to define finite γ-extensions along the directions on which f is increasing, we need to find a bounded convex set C containing O and such that for any cone $K \subseteq K^0$, the hyperplane going through the intersection points of the edges of K with the boundary of this set does not intersect $K \cap P'$. A simplex satisfying those conditions can be easily constructed. Assume that K^0 is spanned by the vectors $u^{01}, u^{02}, \ldots, u^{0n}$. Solve $\max\{\sum_{j=1}^n \lambda_j \mid \sum_{j=1}^n \lambda_j u^{0j} \in K^0 \cap P'\}$ and let Λ^* be the optimal value. Then define the simplex as $\{x \in \mathbb{R}^n \mid x = \sum_{j=1}^n \lambda_j u^{0j}, \sum_{j=1}^n \lambda_j \leq \Lambda, \lambda \geq 0\}$ where $\Lambda > \Lambda^*$.

We are now able to define the γ-extensions: Let $u \neq 0$ be a vector of $\mathbb{R}^n$ and γ be a number satisfying $\gamma \leq f(O)$. Define $\theta = \max\{\alpha \mid f(\alpha u) \geq \gamma,\ \alpha u \in C,\ \alpha \geq 0\}$. The point $y = \theta u$ is called the *γ-extension (of O) along u* (the notion of γ-extension was first introduced by Tuy [11], see, e.g., Horst and Tuy [2]).

Note that if $\gamma = f(O)$ and f is decreasing along u, the γ-extension along u is O. In this paper, γ will always satisfy the condition $\gamma \leq \min\{f(\lambda u) \mid \lambda u \in P', \lambda \geq 0\}$ (recall that O is an interior point of P'), which ensures that the γ-extension will always be distinct from O.

2.3. DELETION TEST

Let $K = \text{cone}\{u^1, u^2, \ldots, u^n\}$ be a cone and $\gamma \leq f(O)$ be a number such that the γ-extensions $y^j = \theta_j u^j$ along $u^j,\ j = 1, 2, \ldots, n$ are distinct from O. Consider the following linear problem

$$LP(P', K, \gamma) \quad \max \sum_{j=1}^n \lambda_j$$
$$\text{s.t.} \quad \begin{cases} A' \sum_{j=1}^n \lambda_j y^j \leq b' \\ \lambda \geq 0. \end{cases}$$

Let $\tilde{\rho}$ be its optimal value. Then if $\tilde{\rho} \leq 1$, we have $x \in conv\{O, y^1, \ldots, y^n\}$, which by quasiconcavity of f implies that $f(x) \geq \gamma$ for all x in $K \cap P'$.

In particular, if γ is the best feasible value for the problem (CP') obtained so far, the cone K can be eliminated from further consideration since no better feasible solution can be found in it.

This deletion test differs from the one originally proposed by Tuy [11] in the addition of the constraint $\lambda \geq 0$ in the definition of $LP(P', K, \gamma)$.

Given an optimal solution $\tilde{\lambda}$ of problem $LP(P', K, \gamma)$, we return to the x-space by defining

$$\tilde{\omega} = \sum_{j=1}^{n} \tilde{\lambda}_j y^j. \tag{1}$$

This point of $K \cap P'$ is used in the subdivision process (see Section 2.4) and in the update of the best known solution (see Section 3). We also define the hyperplane $\tilde{H} = \{x \in \mathbb{R}^n \mid x = \sum_{j=1}^{n} \lambda_j y^j,\ \sum_{j=1}^{n} \lambda_j = \tilde{\rho}\}$ and note by $\tilde{\alpha} x = 1$ its equation.

Now, following [5], consider the dual of $LP(P', K, \gamma)$:

$$DLP(P', K, \gamma) \quad \min \sum_{i=1}^{m} \mu_i b'_i$$
$$\text{s.t.} \quad \begin{cases} \sum_{i=1}^{m} \mu_i a^{'i} y^j \geq 1, & j = 1, \dots, n \\ \mu \geq 0 \end{cases}$$

where $a^{'i}$ denotes the ith row of A', ($i = 1, \dots, m$).

Due to the results in linear programming duality (see, e.g., Luenberger [7]), its optimal value is the same than for $LP(P', K, \gamma)$, that is $\tilde{\rho}$. Let $\hat{\mu}$ be an optimal solution. Define

$$\hat{\alpha} = \frac{1}{\tilde{\rho}} \sum_{i=1}^{m} \hat{\mu}_i a^{'i} \tag{2}$$

and let $\hat{H}$ be the hyperplane of equation $\hat{\alpha} x = 1$. We have the following properties.

PROPOSITION 1. *Let $\tilde{\lambda}$ be an optimal solution of problem $LP(P', K, \gamma)$. Then*

$$\hat{\alpha}(\tilde{\rho} y^j) \geq 1, \quad j = 1, \dots, n$$

with equality for all j such that $\tilde{\lambda}_j > 0$.

Proof. The inequalities are a direct consequence of (2) and of the feasibility of $\hat{\mu}$. Moreover, using the complementary slackness conditions, we have

$$\tilde{\lambda}_j \left(\sum_{i=1}^{m} \hat{\mu}_i a^{'i} y^j - 1 \right) = 0, \qquad j = 1, \dots, n$$

which concludes the proof. □

PROPOSITION 2 [5, Proposition 1]. *The hyperplane $\hat{H}$ supports the polyhedron P' at point $\tilde{\omega}$.*

Proof. Let $x \in P'$. Then $a'^i x \leq b'_i$ for $i = 1, \ldots, m$. Multiplying by $\hat{\mu}_i$ (which are nonnegative) and summing, we obtain

$$\sum_{i=1}^{m} \hat{\mu}_i a'^i x \leq \sum_{i=1}^{m} \hat{\mu}_i b'_i = \tilde{\rho}.$$

Using (2) we get $\hat{\alpha} x \leq 1$, which shows that P' is included in the halfspace $\{x \in \mathbb{R}^n \mid \hat{\alpha} x \leq 1\}$. Now by Proposition 1, $\tilde{\lambda}_j \hat{\alpha}(\tilde{\rho} y^j) = \tilde{\lambda}_j$ for $j = 1, \ldots, n$. Since $\sum_{j=1}^{n} \tilde{\lambda}_j = \tilde{\rho}$ and by (1), we deduce $\hat{\alpha}\tilde{\omega} = 1$. $\square$

COROLLARY 3. *There exists $M > 0$ such that $\|\hat{\alpha}\| \leq M$.*

Proof. Since $\hat{H} = \{x \in \mathbb{R}^n \mid \hat{\alpha} x = 1\}$ supports P', the distance $d(O, \hat{H}) = 1/\|\hat{\alpha}\|$ from O to the hyperplane $\hat{H}$ is bounded from below by the distance $d(O, \delta P')$ from O to the boundary of P'. This distance is nonnull because O is an interior point of P'. Choose $M = 1/d(O, \delta P')$. $\square$

2.4. ω-SUBDIVISION

Let $K = \text{cone}\{y^1, \ldots, y^n\}$ be a cone to be subdivided and let $\tilde{\omega}$ be the point associated with K obtained after the application of the deletion test:

$$\tilde{\omega} = \sum_{j=1}^{n} \tilde{\lambda}_j y^j, \quad \tilde{\lambda} \geq 0.$$

Let $J = \{j \mid \tilde{\lambda}_j > 0\}$. For each $j \in J$ define the cone K^j as the cone K in which the jth edge is replaced by the halfline $[O\tilde{\omega})$. The cones K^j $(j \in J)$ are called *subcones* of K. It is easy to see that they form a partition of the cone K (see, for example, Horst and Tuy [2]). This subdivision is referred to as an ω-subdivision.

Note that $\hat{\alpha} y^j = 1/\tilde{\rho}$ for all j in J by Proposition 1, where $\hat{\alpha}$ and $\tilde{\rho}$ were defined in the previous subsection.

3. Cone partitioning algorithm

We now present a cone partitioning algorithm that is very close to Bali's modification [1] of the original algorithm of Tuy [11]. We show in the next section that this algorithm provides an ε-optimal solution of problem (CP') for any $\varepsilon \geq 0$, i.e., a point $\overline{x} \in K^0 \cap P'$ such that $f(\overline{x}) \leq f^* + \varepsilon$ where f^* is the exact optimal value of problem (CP').

CPω algorithm (cone partitioning via ω-subdivision)

Step 1 (initialization): initialize the incumbent value $\overline{f}$ and solution $\overline{x}$ with the best point among O and the intersection points of the edges of K^0 with the boundary $\delta P'$ of P'. Solve $LP(P', K^0, \overline{f} - \varepsilon)$ obtaining an optimal value $\tilde{\rho}(K^0)$ and a point $\tilde{\omega}(K^0)$. If $\tilde{\rho}(K^0) \leq 1$, stop: $\overline{x}$ is an ε-optimal solution of problem (CP'). Otherwise, set $\mathcal{L} = \{K^0\}$.

Step 2 (subdivision): let $K^* \in \arg\max\{\tilde{\rho}(K) \,|\, K \in \mathcal{L}\}$. Subdivide K^* via the point $\tilde{\omega}(K^*)$ as indicated in Section 2.4. Let $\mathcal{P}$ be the set of subcones.

Step 3 (deletion test): for each cone $K \in \mathcal{P}$, solve $LP(P', K, \overline{f} - \varepsilon)$ obtaining an optimal value $\tilde{\rho}(K)$ and a point $\tilde{\omega}(K)$. If $\tilde{\rho}(K) > 1$, add K to $\mathcal{L}$.

Step 4 (updating the incumbent): if for some $K \in \mathcal{P}$, $f(\tilde{\omega}(K)) < \overline{f}$ then set $\overline{f} \leftarrow f(\tilde{\omega}(K))$; $\overline{x} \leftarrow \tilde{\omega}(K)$.

Step 5 (optimality test): set $\mathcal{L} \leftarrow \mathcal{L} \backslash \{K^*\}$. If $\mathcal{L}$ is empty, stop: $\overline{x}$ is an ε-optimal solution of problem (CP'). Otherwise return to Step 2.

The main difference with Bali's algorithm is that at each iteration we only subdivide the cone with largest $\tilde{\rho}$ instead of all cones of the $\mathcal{L}$ list. Bali's algorithm itself differs from Tuy's covering algorithm by the addition of the constraint $\lambda \geq 0$ in the linear program $LP(P', K, \gamma)$.

4. Convergence

In [12], Tuy has shown that the boundedness of the sequence of generated vectors $\tilde{\alpha}$ would imply the convergence of algorithm CPω. Unfortunately, the hyperplane $\tilde{H}$ may tend to contain entirely $K \cap P'$ as K tends to a degenerated cone, in which case $\|\tilde{\alpha}\|$ would not be bounded (see Jaumard and Meyer [5]). In this section, we show the convergence of algorithm CPω by reasoning on vector $\hat{\alpha}$ instead of $\tilde{\alpha}$.

Before stating the main Theorem, we prove the following result.

PROPOSITION 4. *Let $K = cone\{y^1, \ldots, y^n\}$ be a cone where y^j, $j = 1, \ldots, n$ are γ-extensions for some value γ. Let $\tilde{\omega}$ be the point of $K \cap P'$ and $\hat{H} = \{x \in \mathbb{R}^n \mid \hat{\alpha}x = 1\}$ be the hyperplane supporting P' that correspond respectively to an optimal solution of problems $LP(P', K, \gamma)$ and $DLP(P', K, \gamma)$, and finally let $\tilde{\rho}$ be the common optimal value of these two problems.*

Let $K' = cone\{y'^1, \ldots, y'^n\}$, γ', $\tilde{\omega}'$, $\hat{H}' = \{x \in \mathbb{R}^n \mid \hat{\alpha}'x = 1\}$ and $\tilde{\rho}'$ be defined similarly.

If $K' \subseteq K$ and $\gamma' \leq \gamma$, then

$$1 \geq \hat{\alpha}\tilde{\omega}' \geq \frac{\tilde{\rho}'}{\tilde{\rho}}.$$

Proof. Let $\hat{y}^j$ (respectively $\hat{y}'^j$) be the intersection point of the jth edge of K (respectively of K') $(j = 1, 2, \ldots, n)$ with the hyperplane $\hat{H}_y = \{x \in \mathbb{R}^n \,|\hat{\alpha}x = 1/\tilde{\rho}\}$.

Since $\hat{\alpha} y^j \geq \dfrac{1}{\tilde{\rho}} = \hat{\alpha} \hat{y}^j$, $j = 1, 2, \ldots, n$ by Proposition 1 and definition of the $\hat{y}^j$, we have $\hat{y}^j \in [O y^j]$ for $j = 1, 2, \ldots, n$. Hence, $f(\hat{y}^j) \geq \min\{f(O), f(y^j)\} \geq \gamma$ by quasiconcavity of f, and $\hat{y}^j \in C$ for $j = 1, 2, \ldots, n$. By definition $\hat{y}'^j \in \hat{H}_y \cap K' \subseteq \hat{H}_y \cap K = \text{conv}\{\hat{y}^1, \hat{y}^2, \ldots, \hat{y}^n\}$, hence $f(\hat{y}'^j) \geq \min_{x \in \hat{H}_y \cap K} f(x) \geq \gamma$ and $\hat{y}'^j \in C$ for $j = 1, 2, \ldots, n$.

Recall that y'^j is the γ'-extension along the jth edge of K'. We distinguish between two cases depending on whether $f(y'^j) = \gamma'$ or not. In the first case, since $f(\hat{y}'^j) \geq \gamma \geq \gamma'$, there exists $\beta_j \geq 1$ such that $y'^j = \beta_j \hat{y}'^j$. In the second case, $y'^j \in \delta C$ and $\hat{y}'^j \in C$, hence again there exists $\beta_j \geq 1$ such that $y'^j = \beta_j \hat{y}'^j$. By definition of $\tilde{\omega}'$, we have

$$\tilde{\omega}' = \sum_{j=1}^{n} \tilde{\lambda}'_j y'^j, \qquad \sum_{j=1}^{n} \tilde{\lambda}'_j = \tilde{\rho}', \quad \tilde{\lambda}' \geq 0$$

where $\tilde{\lambda}'$ is an optimal solution of $LP(P', K', \gamma')$. It follows that

$$\tilde{\omega}' = \sum_{j=1}^{n} \tilde{\lambda}'_j \beta_j \hat{y}'^j \quad \text{with } \beta_j \geq 1 \text{ for } j = 1, 2, \ldots, n.$$

Since $\hat{\alpha} \hat{y}'^j = 1/\tilde{\rho}$ by definition for all j, it follows that

$$\hat{\alpha} \tilde{\omega}' = \frac{1}{\tilde{\rho}} \sum_{j=1}^{n} \tilde{\lambda}'_j \beta_j \geq \frac{\tilde{\rho}'}{\tilde{\rho}}.$$

Finally, since the hyperplane $\hat{H} = \{x \in \mathbb{R}^n \mid \hat{\alpha} x = 1\}$ supports P' and $\tilde{\omega}' \in P'$, we have also $\hat{\alpha} \tilde{\omega}' \leq 1$. □

Our main result is the following.

THEOREM 5. *The CPω algorithm is correct and can be infinite only if $\varepsilon = 0$. In this latter case, any cluster point $\overline{\omega}$ of the sequence $\{\tilde{\omega}\}$ generated by the algorithm is an optimal solution of problem (CP').*

Note that this result is stronger than that given in [4], where it was shown that *at least* one cluster point of the sequence $\{\tilde{\omega}\}$ is an optimal solution (in fact this result was expressed by saying that every cluster point of the sequence $\{\overline{x}\}$ is a global minimizer).

In order to prove Theorem 5, denote by K^k the cone selected at Step 2 of iteration k and let $\tilde{\rho}^k = \tilde{\rho}(K^k)$ and $\tilde{\omega}^k = \tilde{\omega}(K^k)$. In addition, let $\hat{H}^k = \hat{H}(K^k) = \{x \in \mathbb{R}^n \mid \hat{\alpha}^k x = 1\}$ be the hyperplane supporting P' associated with K^k, γ^k be the value of the best known solution used to compute the γ-extensions y^{jk}, $j = 1, \ldots, n$ that define the linear program solved for cone K^k, and $\overline{f}^k = f(\overline{x}^k)$ the value of the best known solution at iteration k. If the algorithm stops at iteration N, $\overline{x}^N$ is an ε-optimal solution of problem (CP') since $f(x) \geq \overline{f}^N - \varepsilon$ for all x in $K^0 \cap P'$.

Hence, assume that the algorithm is infinite. Since $\tilde{\omega}^k \in K^0 \cap P'$, $y^{jk} \in C$ for $j = 1, 2, \ldots, n$ and $O \in \operatorname{int} P'$, the sequences $\{\tilde{\omega}^k\}$, $\{y^{jk}\}_k$, $j = 1, 2, \ldots, n$ and $\{\hat{\alpha}^k\}$ are bounded (for the boundedness of $\{\hat{\alpha}^k\}$, see Corollary 3). Since the sequence $\{\overline{f}^k\}$ is nonincreasing and bounded from below by $\min_{x \in K^0 \cap P'} f(x)$, it converges to a limit $\overline{f}^*$. On the other hand, taking into account the selection rule of Step 2 and that $\tilde{\rho}$ does not increase when going from a cone to one of its subcones (see Proposition 4), we obtain that the sequence $\{\tilde{\rho}^k\}$ is nonincreasing. Moreover as it is bounded from below by 1 (because cones satisfying $\tilde{\rho}^k \leq 1$ are eliminated at Step 4), it goes to a limit $\rho^* \geq 1$.

PROPOSITION 6. *The CPω algorithm can be infinite only if* $\varepsilon = 0$. *Furthermore we have* $\rho^* = 1$ *and* $f(\overline{\omega}) = \overline{f}^*$.

Proof. Let $\{\tilde{\omega}^{k_r}\}$ be a subsequence of $\{\tilde{\omega}^k\}$ converging to $\overline{\omega}$. Since each cone is subdivided into a finite number of subcones, there is at least one sequence $\{K^q\}$ of nested cones (i.e., satisfying $K^{q+1} \subseteq K^q$ for all q) such that $\{q\}$ is a subsequence of $\{k_r\}$. Complete $\{q\}$ to obtain a sequence $\{h\}$ such that K^{h+1} is a subcone of K^h for all h. Then $\{q\}$ is a subsequence $\{h_s\}$ of $\{h\}$ and $\overline{\omega} = \lim_{h_s \to \infty} \tilde{\omega}^{h_s}$. Denote by i_h the index of the edge of K^h replaced by the halfline passing through $\tilde{\omega}^h$ in the subdivision procedure. Note that $\hat{\alpha}^h \tilde{\rho}^h y^{i_h h} = 1$ by definition of the subdivision procedure and Proposition 1. Let i_0 be an integer such that $i_{h_s} = i_0$ for infinitely many h_s. Let $\{h_t\} = \{h \mid i_h = i_0\}$. We have $f(y^{i_0 h_{t+1}}) = \gamma^{h_{t+1}} - \varepsilon \leq \overline{f}^{h_t} - \varepsilon$ or $y^{i_0 h_{t+1}} \in \delta C$, and $y^{i_0 h_{t+1}} = \theta_t \tilde{\omega}^{h_t}$ with $\theta_t \geq 1$ for all t. Moreover, $\hat{\alpha}^{h_{t+1}} \tilde{\rho}^{h_{t+1}} y^{i_0 h_{t+1}} = 1$, i.e.,

$$\hat{\alpha}^{h_{t+1}} \tilde{\rho}^{h_{t+1}} \theta_t \tilde{\omega}^{h_t} = 1. \tag{3}$$

Let $\{t_u\}$ be a subsequence of $\{t\}$ such that $\tilde{\omega}^{h_{t_u}} \to \overline{\omega}$, $\theta_{t_u} \to \overline{\theta}$ and $\hat{\alpha}^{h_{t_u+1}} \to \hat{\alpha}$ (i.e., $\{h_{t_u}\}$ is a subsequence of $\{h_s\}$).

By Proposition 4, since $K^{h_{t_u}} \subseteq K^{h_{t_{u-1}+1}}$, we have

$$1 \geq \hat{\alpha}^{h_{t_{u-1}+1}} \tilde{\omega}^{h_{t_u}} \geq \frac{\tilde{\rho}^{h_{t_u}}}{\tilde{\rho}^{h_{t_{u-1}+1}}}. \tag{4}$$

Taking the limit in (3) and (4), and since $\tilde{\rho}^h \to \rho^*$, we obtain $\hat{\alpha}\rho^*\overline{\theta}\overline{\omega} = 1 = \hat{\alpha}\overline{\omega}$ which shows that $\rho^*\overline{\theta} = 1$. Since ρ^* and $\overline{\theta}$ are both greater than or equal to 1, we deduce $\rho^* = \overline{\theta} = 1$. By continuity of f, we have then that $f(\overline{\omega}) \leq \overline{f}^* - \varepsilon$ (note that $\overline{\omega} \in \delta C$ is impossible by definition of the convex set C). But we have also $f(\tilde{\omega}^h) \geq \overline{f}^h$, which implies $f(\overline{\omega}) \geq \overline{f}^*$. This is possible only if $\varepsilon = 0$, in which case $f(\overline{\omega}) = \overline{f}^*$. □

We are now able to prove Theorem 5.

Proof of Theorem 5. We have already shown that if the algorithm is infinite, at least an infinite sequence of cones is generated by the partitioning procedure and that this is possible only if $\varepsilon = 0$. Moreover, the sequence $\{\tilde{\rho}^k\}$ converges to 1 and the sequence $\{\overline{f}^k\}$ to $\overline{f}^* = f(\overline{x})$. It remains to show that $\overline{f}^*$ is the optimal value of problem (CP').

Assume by contradiction that there exists $x' \in K^0 \cap P'$ such that $f(x') < \overline{f}^*$. Let $\{K^{k_h}\}$ be a sequence of cones generated by the subdivision procedure, that contain x' (note that there may be several such sequences if x' belongs to a face of a cone). The sequence $\{K^{k_h}\}$ is infinite. Indeed, if it were finite, let K^{k_N} be the last cone containing x'. There are two possibilities: either K^{k_N} is never selected to be subdivided, in which case $\tilde{\rho}^k \geq \tilde{\rho}^{k_N} > 1$ for $k \geq k_N$, in contradiction with the fact that $\tilde{\rho}^k$ tends to 1. Or K^{k_N} is eliminated, which implies that $f(x') \geq \min_{x \in K^{k_N} \cap P'} f(x) \geq \overline{f}^{k_N} \geq \overline{f}^*$, in contradiction with the assumption $f(x') < \overline{f}^*$.

Let $\hat{y}'^{k_h}$ be the intersection of Ox' with the hyperplane $\hat{H}_y^{k_h} = \{x \in \mathbb{R}^n \mid \hat{\alpha}^{k_h} x = 1/\tilde{\rho}^{k_h}\}$: then $f(\hat{y}'^{k_h}) \geq \gamma^{k_h}$ as $\hat{y}'^{k_h}$ belongs to $K^{k_h} \cap \hat{H}_y^{k_h} = conv\{y^{1k_h}, \ldots, y^{nk_h}\}$ where y^{jk_h}, $j = 1, \ldots, n$ are the γ^{k_h}-extensions. Let y'^* be the $\overline{f}^*$-extension of x' (note that $y'^* \in [Ox']$ with $x' \in \operatorname{int} C$). Since $f(\hat{y}'^{k_h}) \geq \gamma^{k_h} \geq \overline{f}^* = f(y'^*) > f(x')$ and by quasiconcavity of f, we have $\|x'\| > \|y'^*\| \geq \|\hat{y}'^{k_h}\|$. Moreover, since $x' \in K^{k_h} \cap P'$, we have $\|x'\| \leq \tilde{\rho}^{k_h} \|\hat{y}'^{k_h}\|$. Hence,

$$\tilde{\rho}^{k_h} \geq \frac{\|x'\|}{\|\hat{y}'^{k_h}\|} \geq \frac{\|x'\|}{\|y'^*\|} > 1.$$

But $\{\tilde{\rho}^{k_h}\}$ tends to 1 as a subsequence of $\{\tilde{\rho}^k\}$, hence a contradiction. We conclude that there cannot exist $x' \in P'$ such that $f(x') < \overline{f}^*$, and hence that $\overline{f}^*$ is an optimal value of problem (CP'). □

This convergence result can be easily extended to the branch-and-bound variant of this algorithm (see [4]).

5. Conclusion

In this paper, we have given a simplified proof of the convergence of the cone partitioning algorithm with ω-subdivision. This new proof benefits from the insight gained on a new class of hyperplanes and its associated cut/lower bound, developed in an other paper [5]. It is shown that the cone partitioning algorithm is finite when ε-optimal solution are looked for with $\varepsilon > 0$, but only infinite convergence could be shown in the case where $\varepsilon = 0$. No example is known in which the algorithm is infinite, therefore the true status of the cone partitioning algorithm is still an open question when an exact optimal solution is sought.

We heard recently about an independent proof of the result given in this paper for the case $\varepsilon > 0$ by Locatelli [6].

Acknowledgments

Work of the first author was supported by FCAR (Fonds pour la Formation de Chercheurs et l'Aide à la Recherche) grant 95-ER-1048 and by NSERC (Natural Sciences and Engineering Research Council of Canada) grant GP0036426. Work of the second author was supported by NSERC-network grant NET0200815.

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